Structure of Two-qubit Symmetric Informationally Complete POVMs

Huangjun Zhu,[1,2] Yong Siah Teo,[1,2] and Berthold-Georg Englert[1,3]

[1] *Centre for Quantum Technologies, National University of Singapore, Singapore 117543, Singapore*
[2] *NUS Graduate School for Integrative Sciences and Engineering, Singapore 117597, Singapore*
[3] *Department of Physics, National University of Singapore, Singapore 117542, Singapore*

(Dated: August 9, 2010)

In the four-dimensional Hilbert space, there exist 16 Heisenberg–Weyl (HW) covariant symmetric informationally complete positive operator valued measures (SIC POVMs) consisting of 256 fiducial states on a single orbit of the Clifford group. We explore the structure of these SIC POVMs by studying the symmetry transformations within a given SIC POVM and among different SIC POVMs. Furthermore, we find 16 additional SIC POVMs by a regrouping of the 256 fiducial states, and show that they are unitarily equivalent to the original 16 SIC POVMs by establishing an explicit unitary transformation. We then reveal the additional structure of these SIC POVMs when the four-dimensional Hilbert space is taken as the tensor product of two qubit Hilbert spaces. In particular, when either the standard product basis or the Bell basis are chosen as the defining basis of the HW group, in eight of the 16 HW covariant SIC POVMs, all fiducial states have the same concurrence of $\sqrt{2/5}$. These SIC POVMs are particularly appealing for an experimental implementation, since all fiducial states can be connected to each other with just local unitary transformations. In addition, we introduce a concise representation of the fiducial states with the aid of a suitable tabular arrangement of their parameters.

PACS numbers: 03.65.-w, 03.65.Wj, 03.67.-a, 02.10.De

I. INTRODUCTION

A positive operator valued measure (POVM) consists of a set of outcomes represented mathematically as a set of positive operators that sum up to the identity. An *informationally complete* (IC) POVM allows us to reconstruct any quantum state from the probabilities of outcomes. An IC POVM contains at least d^2 outcomes in a d-dimensional Hilbert space. A *minimal* IC POVM contains exactly d^2 outcomes.

A *symmetric informationally complete* (SIC) POVM [1–4], which consists of d^2 pure subnormalized projectors with equal pairwise fidelity, stands out as a fiducial POVM due to its high symmetry and high tomographic efficiency [2, 3, 5, 6]. It is generally believed that SIC POVMs exist in any Hilbert spaces of finite dimensions since Zauner's conjecture [1], although a rigorous mathematical proof is not known. Up to now, analytical solutions have been found in dimensions 2, 3 [7]; 4, 5 [1]; 6 [8]; 7 [3]; 8 [9, 10]; 9–15 [4, 10–13]; 19 [3]; 24, 35, 48 [4]. Numerical solutions with high precision have also been obtained up to $d = 67$ [2, 4].

The interest in SIC POVMs extends well beyond their application in quantum state tomography. The relation between SIC POVMs and mutually unbiased bases (MUB) is another focus of ongoing efforts [14–16]. In the mathematical community, SIC POVMs are studied under the name of equiangular lines [17] and as minimal 2-designs [2]. The Lie algebraic significance of SIC POVMs was also explored recently [18].

A *group-covariant* SIC POVM can be generated from a single reference state—*a fiducial state*—with transformations from a unitary group. Most known SIC POVMs are covariant with respect to the *Heisenberg–Weyl (HW) group* or generalized Pauli group [1–3]. The structure of the HW covariant SIC POVMs can be studied with the aid of the normalizer of the HW group—the *Clifford group* (or the *extended Clifford group* when antiunitary operations are also included), which divides the fiducial states and SIC POVMs into disjoint orbits [3, 19]. SIC POVMs on the same orbit of the extended Clifford group are unitarily or antiunitarily *equivalent* in the sense that they can be transformed into each other with unitary or antiunitary operations [3]. The equivalence relation among SIC POVMs on different orbits is still an open problem for arbitrary dimensions. Recently this problem was solved for prime dimensions [20].

In this paper, we focus on HW covariant SIC POVMs in the four-dimensional Hilbert space which exhibit remarkable additional symmetry beyond what is reflected in the name. According to the numerical calculations by Renes *et al.* [2] as well as by Scott and Grassl [4], there exists a single orbit of 256 fiducial states, constituting 16 SIC POVMs. We shall characterize these fiducial states and SIC POVMs by studying the symmetry transformations within a given SIC POVM and among different SIC POVMs. The symmetry group of each SIC POVM is shown to be a subgroup of the Clifford group, thereby extending recent results on prime dimensions [20]. Furthermore, we find 16 additional SIC POVMs by a regrouping of the 256 fiducial states, and show that they are unitarily equivalent to the original 16 SIC POVMs by establishing an explicit unitary transformation. These additional SIC POVMs from a regrouping of fiducial states have also been noticed by Grassl [12].

We then reveal the additional structure of these SIC POVMs when the four-dimensional Hilbert space is taken as the tensor product of two qubit Hilbert spaces.

A concise representation of the fiducial states is introduced in terms of the generalized Bloch vectors, which allows us to explore the intriguing symmetry of the two-qubit SIC POVMs. In particular, when either the standard product basis or the Bell basis is chosen as the defining basis of the HW group, in eight of the 16 HW covariant SIC POVMs, all the fiducial states have the same concurrence of $\sqrt{2/5}$; hence these fiducial states can be turned into each other with just local unitary transformations. These SIC POVMs are particularly appealing for an experimental implementation, because local unitary transformations are much easier to realize than global ones.

The paper is organized as follows. In Sec. II, we set the stage by recalling basic properties of SIC POVMs and Clifford groups. In Sec. III, we then study the structure of SIC POVMs in the four-dimensional Hilbert space, and construct the 16 additional SIC POVMs by a regrouping of the fiducial states. In Sec. IV, we deal with the structure of two-qubit SIC POVMs. We conclude with a summary.

II. SETTING THE STAGE

A SIC POVM [1–4], $\sum_{j=1}^{d^2}\Pi_j = I$ is composed of d^2 outcomes that are subnormalized projectors, $\Pi_j = |\psi_j\rangle\frac{1}{d}\langle\psi_j|$, such that

$$|\langle\psi_j|\psi_k\rangle|^2 = \frac{1+d\delta_{jk}}{d+1}. \tag{1}$$

The symmetry group $\mathrm{G_{sym}}$ of a SIC POVM consists of all unitary operations that leave the SIC POVM invariant, that is, permute the set of outcomes Π_j. Likewise, the extended symmetry group $\mathrm{EG_{sym}}$ is the larger group that contains also antiunitary operations. A group-covariant SIC POVM is one which can be generated from a fiducial state with a group of unitary operations, such as the symmetry group of the SIC POVM.

Since operators which differ only by overall phase factors implement the same transformation, it is often more convenient to work with the projective version of the symmetry group and extended symmetry group, which are defined as $\overline{\mathrm{G}}_{\mathrm{sym}} = \mathrm{G_{sym}}/I(d)$, $\overline{\mathrm{EG}}_{\mathrm{sym}} = \mathrm{EG_{sym}}/I(d)$, where $I(d)$ is the group consisting of operators which are proportional to the identity operator I. Similarly, throughout the paper, for any unitary group G, $\overline{\mathrm{G}}$ is used to denote the group obtained from G by identifying elements which differ only by overall phase factors.

Almost all known SIC POVMs are covariant with respect to the Heisenberg-Weyl (HW) group or generalized Pauli group D. HW group is generated by the phase operator Z and the cyclic shift operator X defined by their action on the kets $|e_r\rangle$ of the "computational basis":

$$\begin{aligned} Z|e_r\rangle &= \omega^r|e_r\rangle, \\ X|e_r\rangle &= \begin{cases} |e_{r+1}\rangle & r=0,1,\ldots,d-2, \\ |e_0\rangle & r=d-1, \end{cases} \\ D_{p_1,p_2} &= \tau^{p_1p_2}X^{p_1}Z^{p_2}, \end{aligned} \tag{2}$$

where $\omega = e^{2\pi \mathrm{i}/d}, \tau = -e^{\pi\mathrm{i}/d}$, $p_1, p_2 \in Z_d$, and Z_d is the additive group of integer modulo d. The phase factor of D_{p_1,p_2} has been chosen following Appleby [3] to simplify the following discussion. As a consequence of Eq. (1), a fiducial ket $|\psi\rangle$ of the HW group obeys

$$|\langle\psi|D_{p_1,p_2}|\psi\rangle| = \frac{1}{\sqrt{d+1}} \tag{3}$$

for all $(p_1,p_2)\neq(0,0)$, which are d^2-1 equations.

The Clifford group $\mathrm{C}(d)$ is the normalizer of the HW group that consists of unitary operators. Likewise, the extended Clifford group $\mathrm{EC}(d)$ is the larger group that contains also anti-unitary operators. For any operator U in the extended Clifford group, $U|\psi\rangle$ is a fiducial ket whenever $|\psi\rangle$ is one. Fiducial states and SIC POVMs form disjoint orbits under the action of the extended Clifford group. SIC POVMs on the same orbit of the extended Clifford group are unitarily or antiunitarily equivalent in the sense that they can be transformed into each other with unitary or antiunitary operations. The equivalence problem of SIC POVMs among different orbits is closely related to the problem of whether the symmetry group of each HW covariant SIC POVM is a subgroup of the Clifford group. The two problems have been solved for all prime dimensions [20], but remains largely open for non-prime dimensions. For $d = 4$, there is no actual equivalence problem, since there is only one orbit of SIC POVMs; nevertheless we shall give an affirmative answer to the other problem in Sec. III.

To understand the structure of the Clifford group, we need to introduce some additional concepts. Define

$$\bar{d} = \begin{cases} d & \text{if } d \text{ is odd}, \\ 2d & \text{if } d \text{ is even}, \end{cases} \tag{4}$$

and denote by $\mathrm{SL}(2, Z_{\bar{d}})$ the special linear group consisting of 2×2 matrices

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{5}$$

with entries in $Z_{\bar{d}}$ and determinant 1 mod $\bar{d}$. Likewise, $\mathrm{ESL}(2, Z_{\bar{d}})$ is the larger group that contains also the 2×2 matrices with determinant -1 mod $\bar{d}$. $\mathrm{SL}(2, Z_{\bar{d}})\ltimes(Z_d)^2$ is the semidirect product group equipped with the following product rule:

$$(F_1,\chi_1)\circ(F_2,\chi_2) = (F_1F_2, \chi_1+F_1\chi_2), \tag{6}$$

where $F_1, F_2 \in \mathrm{SL}(2, Z_{\bar{d}})$ and $\chi_1,\chi_2\in(Z_d)^2$. Similarly, $\mathrm{ESL}(2, Z_{\bar{d}})\ltimes(Z_d)^2$ is the semidirect product group with the same product rule.

The structure of the Clifford group and the extended Clifford group can best be understood from the following surjective homomorphism given by Appleby [3],

$$\begin{aligned} f_E \ &: \ \mathrm{ESL}(2, Z_{\bar{d}}) \ltimes (Z_d)^2 \to \overline{\mathrm{EC}}(d), \\ &\quad UD_{\mathbf{p}}U^\dagger = \omega^{\langle \chi, F\mathbf{p}\rangle} D_{F\mathbf{p}} \\ \text{for} &\quad U = f_E(F,\chi), \end{aligned} \tag{7}$$

where $\langle \mathbf{p}, \mathbf{q}\rangle = p_2 q_1 - p_1 q_2$. When d is odd, f_E is an isomorphism; when d is even, the kernel contains the following eight elements:

$$\left(\begin{pmatrix} 1+rd & sd \\ td & 1+rd \end{pmatrix}, \begin{pmatrix} sd/2 \\ td/2 \end{pmatrix}\right) \quad \text{for} \quad r,s,t = 0,1. \tag{8}$$

If $\det(F) = 1 \mod \bar{d}$ (see Eq. (5) for the definition of F), and if β is invertible in $Z_{\bar{d}}$, the explicit homomorphism is given by [3]

$$\begin{aligned} (F,\chi) &\to U = D_\chi V_F, \\ V_F &= \frac{1}{\sqrt{d}} \sum_{r,s=0}^{d-1} |e_r\rangle \tau^{\beta^{-1}(\alpha s^2 - 2rs + \delta r^2)} \langle e_s|. \end{aligned} \tag{9}$$

If β is not invertible, there always exists an integer x such that $\delta + x\beta$ is invertible, and F can be written as the product of two matrices $F = F_1 F_2$, where

$$F_1 = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}, \quad F_2 = \begin{pmatrix} \gamma + x\alpha & \delta + x\beta \\ -\alpha & -\beta \end{pmatrix}, \tag{10}$$

such that V_{F_1} and V_{F_2} can be computed according to Eq. (9), then $V_F = V_{F_1} V_{F_2}$ [3].

If $\det(F) = -1$, then $\det(FJ) = 1$ and $(FJ, \chi) \in \mathrm{SL}(2, \bar{d}) \ltimes (Z_d)^2$, where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

Hence the homomorphism images of the elements in $\mathrm{ESL}(2, \bar{d}) \ltimes (Z_d)^2$ can be determined once the images of the elements in $\mathrm{SL}(2,p) \ltimes (Z_p)^2$ and that of $(J, \mathbf{0})$ are determined respectively, where $\mathbf{0}$ is a shorthand for $\binom{0}{0}$. The homomorphism image of $(J, \mathbf{0})$ is the complex conjugation operator $\hat{J}$ [3],

$$\hat{J}: \sum_{r=0}^{d-1} |e_r\rangle a_r \mapsto \sum_{r=0}^{d-1} |e_r\rangle a_r^*, \tag{12}$$

which is clearly basis-dependent (here defined with respect to the computational basis) and has no physical meaning. Following Appleby, $[F, \chi]$ (This is not a commutator!) is used to denote the homomorphism image of (F, χ) throughout the paper.

The rest of the paper focuses on the HW covariant SIC POVMs for $d = 4$ unless otherwise stated.

III. STRUCTURE OF SIC POVMS IN THE FOUR-DIMENSIONAL HILBERT SPACE

For $d = 4$, the order of the Clifford group is 768, and that of the extended Clifford group is 1536. Numerical searches performed by Renes *et al.* [2] as well as by Scott and Grassl [4] suggest that there is only one orbit of fiducial states (both under the Clifford group and the extended Clifford group). In the following discussion, we assume that their numerical searches are exhaustive.

One of the fiducial states is [3] $\rho_{\mathrm{f}} = |\psi_{\mathrm{f}}\rangle\langle\psi_{\mathrm{f}}|$ with

$$\begin{aligned} |\psi_{\mathrm{f}}\rangle &= (|e_0\rangle, |e_1\rangle, |e_2\rangle, |e_3\rangle) \frac{1}{2\sqrt{3+G}} \\ &\times \begin{pmatrix} 1 + e^{-\mathrm{i}\pi/4} \\ e^{\mathrm{i}\pi/4} + \mathrm{i}G^{-3/2} \\ 1 - e^{-\mathrm{i}\pi/4} \\ e^{\mathrm{i}\pi/4} - \mathrm{i}G^{-3/2} \end{pmatrix}, \end{aligned} \tag{13}$$

where $G = (\sqrt{5} - 1)/2$ is the golden ratio. The stability group (within the extended Clifford group) of this fiducial state is the order-6 cyclic group generated by the following antiunitary operator,

$$[A_4, \chi_4] = \left[\begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right] = V\hat{J}, \tag{14}$$

where

$$V \hat{=} \frac{1}{2} \begin{pmatrix} 1 & e^{\mathrm{i}\pi/4} & -1 & e^{\mathrm{i}\pi/4} \\ \mathrm{i} & e^{-3\mathrm{i}\pi/4} & \mathrm{i} & e^{\mathrm{i}\pi/4} \\ 1 & e^{-3\mathrm{i}\pi/4} & -1 & e^{-3\mathrm{i}\pi/4} \\ \mathrm{i} & e^{\mathrm{i}\pi/4} & \mathrm{i} & e^{-3\mathrm{i}\pi/4} \end{pmatrix}, \tag{15}$$

and $\hat{J}$ is the complex conjugation operator defined in Eq. (12). Within the Clifford group, the stability group is generated by $[A_4, \chi_4]^2$. Hence there are 256 fiducial states constituting 16 SIC POVMs on the orbit [2, 3].

A. Symmetry transformations within an HW covariant SIC POVM

In this section, we focus on the symmetry property of a single HW covariant SIC POVM for $d = 4$. In particular, we show that the symmetry group of each HW covariant SIC POVM is a subgroup of the Clifford group, and each HW covariant SIC POVM is covariant with respect to a unique HW group.

Since all SIC POVMs are on the same orbit, it is enough to study the SIC POVM generated from the fiducial state with the ket in Eq. (13) under the action of the HW group. To demonstrate that the symmetry group $\mathrm{G_{sym}}$ (extended symmetry group $\mathrm{EG_{sym}}$) of this SIC POVM is a subgroup of the Clifford group (extended Clifford group), it is enough to show that the stability group of the fiducial state ρ_{f} within the symmetry group is the same as that within the Clifford group, which is generated by $[A_4, \chi_4]^2$.

To simplify the notation in the following discussion, we use the ordered pair (p_1, p_2) to represent the fiducial state with the ket $D_{p_1,p_2}|\psi_{\rm f}\rangle$. Under the action of $[A_4, \chi_4]^2$, the 15 fiducial states other than $\rho_{\rm f} \hat{=} (0,0)$ in the SIC POVM form five orbits:

$$\begin{aligned} O_1 &= \{(1,0),(0,3),(3,1)\}, \\ O_2 &= \{(3,3),(3,2),(2,3)\}, \\ O_3 &= \{(0,1),(1,3),(3,0)\}, \\ O_4 &= \{(1,2),(2,1),(1,1)\}, \\ O_5 &= \{(2,0),(0,2),(2,2)\}. \end{aligned} \tag{16}$$

Now let $\rho_{j_1}, \rho_{j_2}, \rho_{j_3}$ be any triple of different fiducial states in the SIC POVM. Notice that the triple product traces $\mathrm{tr}(\rho_{j_1}\rho_{j_2}\rho_{j_3})$ are invariant under unitary transformations. Hence any unitary transformation in the stability group (within the symmetry group of the SIC POVM) of $\rho_{\rm f}$ must preserve these invariants. However, at least one of these invariants would be violated, if there exists any unitary transformation in the stability group other than that generated by $[A_4, \chi_4]^2$. In conclusion, the symmetry group of each HW covariant SIC POVM is a subgroup of the Clifford group for $d = 4$.

From the previous discussions, the order of the symmetry group $\overline{\mathrm{G}}_{\rm sym}$ (extended symmetry group $\overline{\mathrm{EG}}_{\rm sym}$) of each SIC POVM is 48 (96), which is much smaller than that of the symmetry group of a 15-dimensional regular simplex. Moreover, it is not always possible to transform a pair of fiducial states to another pair with either a unitary or antiunitary operation within the extended symmetry group. Note that the HW group is a normal Sylow 2-subgroup of $\overline{\mathrm{G}}_{\rm sym}$ (for any prime p, a Sylow p-subgroup of a finite group is a subgroup of order p^n such that p^n is the largest power of p that divides the order of the group [21]), hence there is only one Sylow 2-subgroup (here an order-16 subgroup) in $\overline{\mathrm{G}}_{\rm sym}$ according to Sylow's theorem. As a result, each HW covariant SIC POVM is covariant with respect to a unique HW group for $d = 4$. This observation extends the previous result on prime dimensions not equal to three [20].

Within each HW covariant SIC POVM, the triple product traces $\mathrm{tr}(\rho_{j_1}\rho_{j_2}\rho_{j_3})$ may take on 17 different values (eight pairs of conjugates and one real number). As we shall see shortly, there exists a continuous family of inequivalent triples of normalized states with equal pairwise fidelity of 1/5. In most cases, it is impossible to extend such a triple to a full HW covariant SIC POVM, even in principle, in contrast with the situation for $d = 2$ or $d = 3$. This phenomenon is common in Hilbert spaces of higher dimensions.

In a d-dimensional Hilbert space with $d \geq 3$, the three states corresponding to the three kets, respectively, in the following equation have equal pairwise fidelity of $1/(d+1)$:

$$|\varphi_1\rangle \hat{=} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ |\varphi_2\rangle \hat{=} \begin{pmatrix} \frac{1}{\sqrt{d+1}} \\ \frac{\sqrt{d}}{\sqrt{d+1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ |\varphi_3\rangle \hat{=} \begin{pmatrix} \frac{1}{\sqrt{d+1}} \\ u(\theta)e^{i\theta} \\ v(\theta) \\ \vdots \\ 0 \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned} u(\theta) &= \frac{-\cos\theta + \sqrt{(\cos\theta)^2 + d}}{\sqrt{d(d+1)}}, \\ v(\theta) &= \sqrt{\frac{d^2 - d - 2(\cos\theta)^2 + 2\cos\theta\sqrt{(\cos\theta)^2 + d}}{d(d+1)}} \end{aligned} \tag{18}$$

with $-\pi \leq \theta < \pi$. Let

$$\begin{aligned} \phi(\theta) &= \arg\bigl\{\mathrm{tr}[(|\varphi_1\rangle\langle\varphi_1|)(|\varphi_2\rangle\langle\varphi_2|)(|\varphi_3\rangle\langle\varphi_3|)]\bigr\} \\ &= \arg\Bigl[\frac{1 + e^{\mathrm{i}\theta}(-\cos\theta + \sqrt{(\cos\theta)^2 + d})}{(d+1)^2}\Bigr]. \end{aligned} \tag{19}$$

There is a one-to-one correspondence between θ and $\phi(\theta)$, and the angle $\phi(\theta)$ of the triple product may also take on any value between $-\pi$ and π. Since this angle is invariant under unitary transformations, any two different ordered triples of states in the family defined by Eq. (17) are inequivalent. On the other hand, any two ordered triples can be turned into each other by a suitable unitary transformation if the angles of the respective triple product traces are equal.

For $d = 2$, any triple of states is uniquely specified by the pairwise fidelity, up to unitary transformations. And the triple can always be extended to a full SIC POVM if the pairwise fidelity is $1/(d+1)$. For $d = 3$, there exists a continuous family of inequivalent SIC POVMs [1–3]; with a suitable choice of SIC POVMs and fiducial states, the angle of the triple product trace may take on any value between $-\pi$ and π [20]. Hence any triple of states with equal pairwise fidelity of $1/(d+1)$ can be extended to a full HW covariant SIC POVM.

For $d = 4$, as far as the SIC POVM generated from the fiducial state in Eq. (13) is concerned, the angle of the triple product trace of distinct fiducial states may only take on 17 different values. Hence, it is impossible to extend a generic triple of states with equal pairwise fidelity of $1/(d+1)$ to a full HW covariant SIC POVM. The same conclusion also holds for $d > 4$ if there are only a finite number of inequivalent HW covariant SIC POVMs, which seems to be the case according to the numerical searches performed by Scott and Grassl [4].

Similarly, it is reasonable to expect that it is generally impossible to extend a given set of k states ($k < d^2$) with equal pairwise fidelity of $1/(d+1)$ to a full SIC POVM.

This observation implies that we need some global constraints in addition to the local constraint of equal pairwise fidelity to fully characterize the structure of a SIC POVM. It also illustrates the difficulty of constructing a SIC POVM through adding states one by one in a successive manner.

B. Reconstruction of the HW group from a given SIC POVM

Most known examples of SIC POVMs are constructed from fiducial states under the action of the HW group. In this section we investigate the inverse problem: reconstruct the HW group from a given SIC POVM. This problem is relevant in determining whether a SIC POVM constructed with a different method is covariant with respect to the HW group, and in determining the equivalence relation among SIC POVMs, as we shall see in Sec. III D. Our approach is described for SIC POVMs in the four-dimensional Hilbert space; however, it can be generalized to SIC POVMs in Hilbert spaces of other dimensions if certain conditions are satisfied.

Let M be the sum of four different fiducial states in the SIC POVM generated from the fiducial state in Eq. (13), $M = \rho_{j_1} + \rho_{j_2} + \rho_{j_3} + \rho_{j_4}$. Our reconstruction scheme is based on the set of eigenvalues and eigenkets of M. Since the SIC POVM is group-covariant and eigenvalues are invariant under unitary transformations, we can assume that $\rho_{j_1} = \rho_{\mathrm{f}}$ without loss of generality. When the four fiducial states are related by the transformation Z, we have

$$\begin{aligned} M &= \sum_{j=0}^{3} Z^j \rho_{\mathrm{f}} Z^{-j} = \sum_{j=0}^{3} |e_j\rangle \lambda_j \langle e_j|, \\ \lambda_{0,2} &= \frac{1}{\sqrt{5}}(2 \pm \sqrt{2})G, \quad \lambda_{1,3} = \frac{2}{\sqrt{5}G} \pm \sqrt{\frac{2}{5G}}. \end{aligned} \tag{20}$$

If ρ_{f} is replaced by another fiducial state, then the order of the diagonal entries of M may be changed, however, λ_0 and λ_2 never appear in adjacent position in the diagonal of M, neither do λ_1 and λ_3. In addition, since the generator of the stabilizer of ρ_{f} in Eq. (14) implements the following cyclic transformation:

$$Z \to XZ^2 \to XZ^3 \to X^2Z \to X^3 \to XZ \to Z, \tag{21}$$

the set of eigenvalues of M is the same if the four states $\rho_{j_1}, \rho_{j_2}, \rho_{j_3}, \rho_{j_4}$ are related by any order-4 element in the HW group. Further calculation shows that the set of eigenvalues of M will be different if the four states cannot be connected by any order-4 element in the HW group.

We are now ready to reconstruct the HW group from a given SIC POVM based on the previous observations:

1. Find four different fiducial states such that the set of eigenvalues of their sum M is the same as that given in Eq. (20), and calculate the normalized eigenkets $|e'_k\rangle$ of M corresponding to the eigenvalues λ_k for $k = 0, 1, 2, 3$ respectively. Then the operator $Z' = \sum_{k=0}^{3} |e'_k\rangle \mathrm{i}^k \langle e'_k|$ is a generator of the HW group to be reconstructed.
2. Under the action of Z', the 16 fiducial states form four orbits of equal length. Choose four fiducial states, one from each orbit, such that the four states possess the same property as in the first step, then construct another operator X' as in the first step.
3. The group generated by the two operators X', Z' is exactly the HW group of interest.

In addition to reconstructing the HW group from a given HW covariant SIC POVM, the above procedure can also be applied to check the group covariance of a SIC POVM constructed with a different method. If we cannot find the set of fiducial states required in Steps 1 or 2, or the group thus reconstructed is not unitarily equivalent to the HW group, or the SIC POVM is not covariant under the group thus constructed (if such a SIC POVM exists), then the SIC POVM is not HW covariant.

C. Symmetry transformations among HW covariant SIC POVMs

In this section, we study the symmetry transformations among the 16 HW covariant SIC POVMs for $d = 4$ and reveal the structure underlying these SIC POVMs.

To describe the symmetry transformations among the 16 SIC POVMs, we first need to label each SIC POVM with a unique number for later reference. Define $V_n = [F_n, \mathbf{0}]$ for $n = 1, 2, \ldots, 16$, where F_ns are given by

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 \\ 5 & 7 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 6 & 7 \\ 3 & 5 \end{pmatrix}, \\ &\begin{pmatrix} 0 & 3 \\ 5 & 5 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix}, \quad \begin{pmatrix} 6 & 7 \\ 7 & 7 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}, \\ &\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 6 & 7 \\ 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 \\ 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix}, \\ &\begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 7 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 \\ 5 & 0 \end{pmatrix}. \end{aligned} \tag{22}$$

Let the image of the SIC POVM containing the fiducial state in Eq. (13) under the transformation V_n be SIC POVM No. n, then this correspondence between the 16 HW covariant SIC POVMs and the 16 numbers $n = 1, 2, \ldots, 16$ is one-to-one. Here the F_ns have been chosen in foresight to simplify the following discussion.

Since we are now only concerned with the transformations among different SIC POVMs, the groups $\mathrm{G_{SYM}} = \overline{\mathrm{C}}(4)/\overline{D}$ and $\mathrm{EG_{SYM}} = \overline{\mathrm{EC}}(4)/\overline{D}$ properly describe the symmetry operations we consider. As an abstract group, $\mathrm{G_{SYM}}$ is isomorphic to the special linear group $\mathrm{SL}(2,4)$ defined in Sec. II; likewise $\mathrm{EG_{SYM}}$ is isomorphic to the extended special linear group $\mathrm{ESL}(2,4)$. Coincidently,

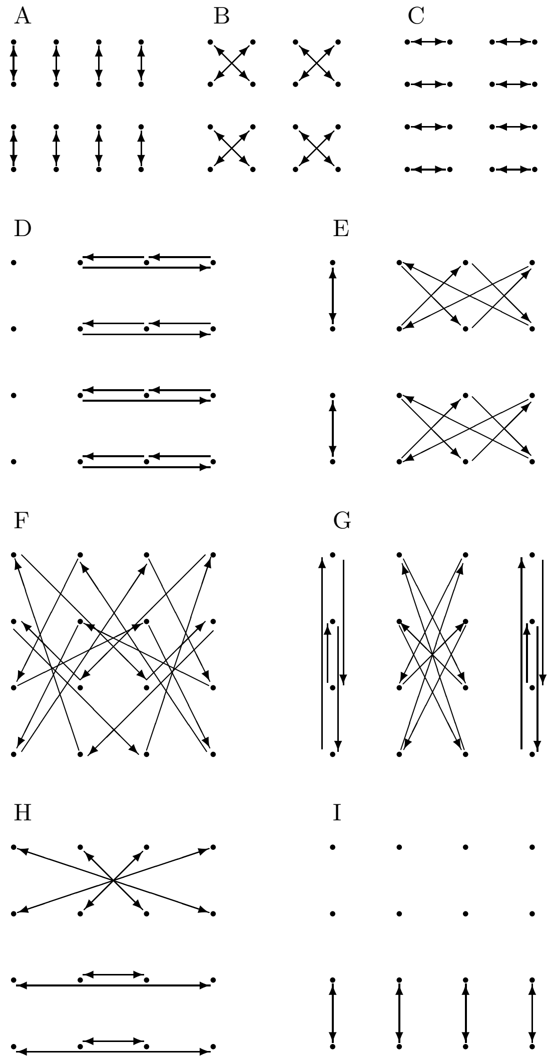

FIG. 1: Illustration of the symmetry transformations among the 16 HW covariant SIC POVMs induced by elements in the group $\mathrm{EG_{SYM}} = \overline{\mathrm{EC}}(d)/\overline{D}$ (see Sec. III C). Here, every dot represents a SIC POVM arranged as in Table I, and every arrow starts from a SIC POVM before the symmetry transformation and ends at another SIC POVM after the symmetry transformation. Only one element in each conjugacy class of $\mathrm{G_{SYM}}$ is chosen as a representative, the transformations induced by other elements within the same conjugacy class can be obtained by permuting the columns. In the case of order-4 elements, only two out of the four conjugacy classes are chosen; the elements in the other two conjugacy classes are the inverses of the elements in these two conjugacy classes respectively, so their transformations can be obtained by reversing the arrows. Plot A: order-2 element in the center of $\mathrm{G_{SYM}}$; plots B and C: another two order-2 elements from the other two conjugacy classes respectively; plots D and E: an order-3 element and an order-6 element respectively; plots F and G: two order-4 elements from two different conjugacy classes respectively; plot H: the complex conjugation operation; plot I: the complex conjugation operation followed by an appropriate order-2 element in $\mathrm{G_{SYM}}$.

TABLE I: Arrangement of the 16 HW covariant SIC POVMs for $d = 4$. Each number n, with $1 \leq n \leq 16$, represents the HW covariant SIC POVM obtained by transforming the SIC POVM containing the fiducial state in Eq. (13) with the unitary transformation $[F_n, \mathbf{0}]$ specified in Eq. (22).

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

the order of $\mathrm{G_{SYM}}$ is the same as the order of the symmetry group $\overline{\mathrm{G}}_{\mathrm{sym}}$ of a single SIC POVM, that is 48; however, the two groups are not isomorphic. The group $\mathrm{G_{SYM}}$ consists of the identity, seven order-2 elements, eight order-3 elements, 24 order-4 elements and eight order-6 elements. There are three conjugacy classes for order-2 elements, with one, three and three elements respectively. There are four conjugacy classes for order-4 elements, each with six elements; elements in two of the classes are the inverses of the elements in the other two classes respectively. There is only one conjugacy class for either order-3 elements or order-6 elements. The center of $\mathrm{G_{SYM}}$ is generated by the order-2 element which has only one conjugate.

If the 16 HW covariant SIC POVMs are arranged in a 4×4 square as in Table I, then the effect of the symmetry transformations of the group $\mathrm{G_{SYM}}$ can be delineated in a pictorial way as shown in Fig. 1. The effect of only one group element in each conjugacy class is shown; the effect of other group elements within the same conjugacy class can be obtained simply by permuting the columns representing the SIC POVMs.

According to Fig. 1, the symmetry transformations among the 16 SIC POVMs can be decomposed into row transformations and column transformations. In addition to the identity, all order-3 elements and one class of order-2 elements (see plots D and C in Fig. 1) transform the SIC POVMs within each row, and with the same effect in every row. They constitute an order-12 normal subgroup of $\mathrm{G_{SYM}}$, which can also be identified as the alternating group of the four columns. The quotient group of $\mathrm{G_{SYM}}$ with respect to this group of row transformations can then be identified with an order-4 cyclic subgroup (generated by the cyclic permutation of the four rows $1 \to 3 \to 2 \to 4 \to 1$, see plots F and G in Fig. 1) of the symmetry group of the four rows. Similarly, the quotient group of $\mathrm{EG_{SYM}}$ can be identified with an order-8 subgroup of the symmetry group of the four rows.

D. Additional SIC POVMs from regrouping of the fiducial states

In this section, we show that there are 16 additional SIC POVMs from a regrouping of the 256 fiducial states. These additional SIC POVMs are quite peculiar in that they are not constructed from fiducial states under the

action of the HW group. Nevertheless, they are unitarily equivalent to the original SIC POVMs as we shall see shortly. These additional SIC POVMs from regrouping fiducial states have been noticed by Grassl, who also showed that they are covariant with respect to the HW group but in a different basis [12].

The construction of these additional SIC POVMs is best illustrated if the 16 original HW covariant SIC POVMs are arranged in a 4×4 square as in Table I. Under the action of the abelian subgroup $H = \{I, X^2, Z^2, X^2Z^2\}$ of the HW group, the 16 fiducial states in each SIC POVM form four orbits of equal size. Given four fiducial states in a SIC POVM connected by H, then in each of the other three SIC POVMs in the same row, there exist exactly four fiducial states which are also connected by H, such that the pairwise fidelity between these four fiducial states and the given four fiducial states is $\frac{1}{5}$—the value required to form a SIC POVM for dimension four. It turns out that the 16 states thus chosen also form a SIC POVM. In this way, four additional SIC POVMs can be constructed by the regrouping of the fiducial states in the four original SIC POVMs in each row, that is, 16 additional SIC POVMs in total. Moreover, inspection of the pairwise fidelity among all 256 fiducial states shows that there are no other SIC POVMs that can be constructed by regrouping these fiducial states.

Surprisingly, these additional SIC POVMs are unitarily equivalent to the original ones, despite the different method of construction. According to the procedure described in Sec. III B, we can reconstruct the HW group D' for these additional SIC POVMs which are generated by the following two operators:

$$X' = \left[\begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \hat{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} Z' &= \left[\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right] \\ &\hat{=} \frac{1}{2}\begin{pmatrix} 0 & 1+\mathrm{i} & 0 & -1+\mathrm{i} \\ 1+\mathrm{i} & 0 & -1+\mathrm{i} & 0 \\ 0 & -1+\mathrm{i} & 0 & 1+\mathrm{i} \\ -1+\mathrm{i} & 0 & 1+\mathrm{i} & 0 \end{pmatrix}. \end{aligned} \tag{23}$$

Note that D' is also a subgroup of the Clifford group of D. Now it is straightforward to verify that the unitary operator

$$U \hat{=} \frac{1}{2}\begin{pmatrix} -\mathrm{i} & -1 & -\mathrm{i} & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ -\mathrm{i} & 1 & -\mathrm{i} & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{pmatrix} \tag{24}$$

transforms the standard HW group D into the HW group D' for these additional SIC POVMs, that is $D' = UDU^\dagger$. Meanwhile, U is also the unitary transformation between the original SIC POVMs and these additional SIC POVMs. The fiducial state ρ_{f} defined in Eq. (13) remains invariant under this transformation due to our specific choice of U.

Further analysis shows that there are 32 subgroups of the Clifford group which are unitarily equivalent to the HW group D (including D itself), out of which only D and D' are normal. The group generated by the Clifford group and U is the normalizer (within the unitary group) of the Clifford group, of which the Clifford group is a subgroup with index 2. This group exhausts all unitary symmetry operations among the 256 fiducial states.

For $d = 3$, there exists a continuous family of orbits of SIC POVMs, and there are 72 fiducial states constituting eight SIC POVMs on each generic orbit [1–3]. 24 additional SIC POVMs can be obtained from a suitable regrouping of the 72 fiducial states. However, these additional SIC POVMs are not equivalent to the original ones [20]. For other dimensions, as far as the SIC POVMs found by Scott and Grassl [4] are concerned, only for the orbits 8b and 12b (according to the labeling scheme of Scott and Grassl), additional SIC POVMs can be obtained by a suitable regrouping of the fiducial states [22]. We are still trying to understand why these dimensions and orbits of SIC POVMs are special in this aspect.

IV. TWO-QUBIT SIC POVMS

In this section we study the additional structure of SIC POVMs when the four-dimensional Hilbert space is perceived as a tensor product of two qubit Hilbert spaces. The appearance of these additional properties are generally basis dependent, because it matters how the four-dimensional Hilbert space is tensor-factored into two two-dimensional spaces. We shall focus on the product basis and the Bell basis in the following discussion, since the additional structure is most appealing in these two specific bases.

Before discussing those properties related to the specific bases, we first mention a result which is basis independent. The average purity of the single qubit reduced states of states in any two-qubit SIC POVM is $4/5$, that is, the average tangle or squared concurrence of states in the two-qubit SIC POVM is $2/5$. More generally, in a bipartite Hilbert space of subsystem dimensions d_1 and d_2 respectively and total dimension $d = d_1 d_2$, the average purity of the reduced states of either party of states in any SIC POVM (if such a SIC POVM exists) is $(d_1 + d_2)/(d_1 d_2 + 1)$—this value is equal to the average over all pure states in the bipartite Hilbert space with respect to the Haar measure [23].

A. Two-qubit SIC POVMs in the product basis

For a single qubit, any state can be expressed in terms of the identity operator I and the three Pauli operators σ_j for $j = x, y, z$; the coefficients of expansion define the

TABLE II: Arrangement of the components of the generalized Bloch vector of each fiducial state.

	r_x	r_y	r_z
s_x	C_{xx}	C_{xy}	C_{xz}
s_y	C_{yx}	C_{yy}	C_{yz}
s_z	C_{zx}	C_{zy}	C_{zz}

Bloch vector. In the case of two qubits, any state ρ can be expressed in terms of the tensor products of the identity and the Pauli operators of each qubit respectively:

$$\rho = \frac{1}{4}\Big(I\otimes I + \sum_{j=x,y,z} r_j I\otimes\sigma_j + \sum_{j=x,y,z} s_j\sigma_j\otimes I + \sum_{j,k=x,y,z} C_{jk}\sigma_j\otimes\sigma_k\Big). \tag{25}$$

Let

$$v = \big(r_x, r_y, r_z, s_x, s_y, s_z, C_{xx}, c_{xy}, C_{xz}, C_{yx}, C_{yy}, C_{yz}, C_{zx}, c_{zy}, C_{zz}\big)^T; \tag{26}$$

in analogy to the case of a single qubit, the vector v will be referred to as the generalized Bloch vector (GBV) of ρ. Although quite common, this terminology is slightly abusive and somewhat misleading. The s column and the three columns of C in Table II transform like three-dimensional column vectors when the first qubit is rotated by local unitary transformations; likewise, the r row and the three rows of C are row vectors for local unitary transformations of the second qubit. In short, the two single-qubit Bloch vectors are *vectors*, and the two-qubit "double vector" C is a *dyadic*.

The structure of the GBVs of the fiducial states are best illustrated if the components are arranged as in Table II. When the standard product basis is chosen as the defining basis of the HW group, that is, $|e_0\rangle = |00\rangle, |e_1\rangle = |01\rangle, |e_2\rangle = |10\rangle, |e_3\rangle = |11\rangle$, the 256 fiducial states divide into two classes, according to the structure of their GBVs. The first class consists of the 128 fiducial states in the first eight HW covariant SIC POVMs, and the second class of the 128 states in the last eight SIC POVMs (according to the labeling scheme described in Sec. III C). The structure of the GBV of each fiducial state in the first class is shown in the top tabular of Table III, where

$$a, b, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 = \pm 1, \quad B = \frac{1}{\sqrt{5}}, \quad A_\pm = \frac{\sqrt{1\pm\sqrt{G}}}{\sqrt{5}}. \tag{27}$$

The eight sign factors $a, b, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ obey the constraint

$$ab\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 = 1. \tag{28}$$

There are seven free sign factors, giving a total of 128 combinations of values, and specifying exactly 128 fiducial states in the first class. In addition, each SIC POVM

TABLE III: The structure of the generalized Bloch vector of each fiducial state in the first class (top) and that in the second class (bottom) when the standard product basis is chosen as the defining basis of the HW group.

	$\beta_1 A_b$	$\beta_2 A_{-b}$	β_3 B
$\alpha_1 B$	$\alpha_1\beta_1 A_{-b}$	$\alpha_1\beta_2 A_b$	$\alpha_1\beta_3 B$
$\alpha_2 A_a$	$\sqrt{2}a\alpha_2\beta_1 A_a\delta_{a,b}$	$\sqrt{2}a\alpha_2\beta_2 A_a\delta_{a,-b}$	$\alpha_2\beta_3 A_{-a}$
$\alpha_3 A_{-a}$	$-\sqrt{2}a\alpha_3\beta_1 A_{-a}\delta_{-a,b}$	$-\sqrt{2}a\alpha_3\beta_2 A_{-a}\delta_{a,b}$	$\alpha_3\beta_3 A_a$

	$\beta_1 A_a$	$\beta_2 A_a$	$\beta_3 B$
$\alpha_1 B$	$\alpha_1\beta_1 A_{-a}$	$\alpha_1\beta_2 A_{-a}$	$\alpha_1\beta_3 B$
$\alpha_2 A_a$	$a^{(1-b)/2}\alpha_2\beta_1 G_{-b}$	$a^{(1+b)/2}\alpha_2\beta_2 G_b$	$\alpha_2\beta_3 A_{-a}$
$\alpha_3 A_a$	$a^{(1+b)/2}\alpha_3\beta_1 G_b$	$a^{(1-b)/2}\alpha_3\beta_2 G_{-b}$	$\alpha_3\beta_3 A_{-a}$

in the first class is specified by the following three sign functions, each taking a constant value for the fiducial states in a given SIC POVM:

$$h_1 = b\alpha_2\alpha_3\beta_3, \quad h_2 = \alpha_1\alpha_2\alpha_3, \quad h_3 = ab\alpha_1. \tag{29}$$

Each combination of the eight sign factors which does not satisfy Eq. (28) specifies a Hermitian operator Q which is not positive semidefinite. Nevertheless, Q can be written as the partial transpose (with respect to the computational basis) of a fiducial state, and satisfies the following 15 equations as each fiducial state does:

$$\mathrm{tr}(QD_{p_1,p_2}QD^\dagger_{p_1,p_2}) = \frac{1}{5} \tag{30}$$

for all $(p_1, p_2) \neq (0, 0)$. These equations mean that the 16 operators generated from Q under the action of the HW group also form a 15-dimensional regular simplex in the Hilbert space of Hermitian operators.

The structure of the GBV of each fiducial state in the second class is shown in the bottom tabular of Table III, where

$$G_\pm = \frac{\sqrt{1\pm G}}{\sqrt{5}} \tag{31}$$

and $A_\pm, B$ are defined in Eq. (27). There is also one constraint among the eight sign factors, namely

$$b\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 = 1. \tag{32}$$

Each SIC POVM in the second class is also specified by three sign functions,

$$h_1 = ab\alpha_1\beta_3, \quad h_2 = -\alpha_1\alpha_2\alpha_3, \quad h_3 = b\alpha_1. \tag{33}$$

When the SIC POVMs are arranged as in Table I and Eq. (22), the sign function h_1 is a constant in each row, while the sign functions h_2, h_3 are constants in each column, see Table IV. This is one of the reasons why the numbering in Table I was done that way.

TABLE IV: The values of the three sign functions h_1, h_2, h_3 (defined in Eqs. (29) and (33)) for each HW covariant SIC POVM labeled according to Sec. III C.

	$h_2 = 1$ $h_3 = -1$	$h_2 = 1$ $h_3 = 1$	$h_2 = -1$ $h_3 = 1$	$h_2 = -1$ $h_3 = -1$
$h_1 = -1$	1	2	3	4
$h_1 = 1$	5	6	7	8
$h_1 = 1$	9	10	11	12
$h_1 = -1$	13	14	15	16

Since the standard product basis is chosen as the defining basis of the HW group, Z and X^2 are both local unitary operators. The 16 fiducial states in each SIC POVM divide into two sets of equal size, such that the eight fiducial states in each set have the same concurrence. For each SIC POVM in the second class, eight fiducial states have concurrence of $\sqrt{(2+2\sqrt{G})/5}$, and the other eight have concurrence of $\sqrt{(2-2\sqrt{G})/5}$. What is peculiar for each SIC POVM in the first class is that all 16 fiducial states have the same concurrence of $\sqrt{2/5}$ (tangle of 2/5). One could say that these symmetric IC POVMs are not just symmetric, they are supersymmetric. This supersymmetry is remarkable, indeed.

Fiducial states in the first class can be turned into each other with just local unitary transformations. This property is particularly appealing for an experimental implementation of these POVMs, because local unitary transformations are much easier to realize than global ones. On the other hand, the eight SIC POVMs in the first class can be transformed into each other with local Clifford unitary transformations, and so can the eight SIC POVMs in the second class.

Since the average tangle of fiducial states in any two-qubit SIC POVM is 2/5, and the concurrence and entanglement of formation are both concave functions of the tangle, the average concurrence or entanglement of formation of fiducial states in a SIC POVM is maximized when the tangle (concurrence) of each fiducial state is the same, as for each SIC POVM in the first class.

Although all fiducial states in each SIC POVM in the first class have the same concurrence, nevertheless, it is impossible to connect all fiducial states with only local unitary transformations from the symmetry group $\overline{\mathrm{G}}_{\mathrm{sym}}$ of the SIC POVM. The same conclusion also holds for any other basis. Suppose otherwise, to connect all fiducial states in a SIC POVM, the order of the local unitary transformation group is necessarily a multiple of 16; on the other hand, the order must divide the order of $\overline{\mathrm{G}}_{\mathrm{sym}}$, which is 48. It follows that the local unitary transformation group must have order either 16 or 48, and thus contains the HW group as a subgroup, since the HW group is the only order-16 subgroup in $\overline{\mathrm{G}}_{\mathrm{sym}}$ according to Sec. III A. However, the HW group cannot be a local unitary group, hence a contradiction would arise.

Furthermore, in each SIC POVM, exactly two fiducial states have the same single-qubit reduced states for the first qubit, and the same is true for the second qubit. The end points of the Bloch vectors of the eight distinct single-qubit reduced states for each qubit form a quite regular pattern, especially for the second qubit and for each SIC POVM in the first class, where they form a cube.

In bipartite Hilbert spaces, SIC POVMs such that all fiducial states have the same Schmidt coefficients are quite rare. In eight-dimensional Hilbert space, there is a SIC POVM which is covariant with respect to an alternative version of the HW group—the three fold tensor product of the Pauli group [9, 10]. Since all fiducial states are connected to each other by a local unitary group, they have the same Schmidt coefficients according to any bipartition of the three parties. As far as the SIC POVMs found by Scott and Grassl [4] are concerned, which are covariant with respect to the HW group defined in Eq. (2), such SIC POVMs only exist on the orbits 4a, 6a, 12b, 28c (according to the labeling scheme of Scott and Grassl). Interestingly, in all these examples, $d_2 = 2$ ($d_1 = d/d_2 = 2, 3, 6, 14$); hence concurrence is well defined. According to the discussion at the beginning of this section, the purity of the reduced density matrix of each fiducial state is $(d_1+2)/(2d_1+1)$, thus the concurrence of each fiducial state is $\sqrt{2(d_1-1)/(2d_1+1)}$. When $d_1 = 2$, this is exactly the concurrence of each fiducial state in the first class of the two-qubit SIC POVM.

B. Two-qubit SIC POVMs in the Bell basis

Now consider the Bell basis as the defining basis of the HW group, that is,

$$\begin{aligned} |e_0\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\ |e_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |e_2\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |e_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned} \tag{34}$$

The structure of the GBV of each fiducial state in the first class (according to the classification scheme in Sec. IV A) is shown in the top tabular of Table V, where $A_\pm, B$ are defined in Eq. (27). As in the case of the product basis, here $a, b, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ may only take on values ±1, and satisfy one constraint,

$$ab\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 = 1. \tag{35}$$

In addition, each SIC POVM is specified by three sign functions,

$$h_1 = -b\alpha_1\beta_1\beta_2\beta_3, \; h_2 = -\beta_1\beta_2\beta_3, \; h_3 = ab\beta_1. \tag{36}$$

TABLE V: The structure of the generalized Bloch vector of each fiducial state in the first class (top) and that in the second class (bottom) when the Bell basis is chosen as the defining basis of the HW group.

	$\beta_1 B$	$\sqrt{2}\beta_2 A_a \delta_{a,b}$	$\sqrt{2}\beta_3 A_{-a}\delta_{-a,b}$
$\alpha_1 B$	$\alpha_1\beta_1 B$	$\sqrt{2}\alpha_1\beta_2 A_{-a}\delta_{a,b}$	$\sqrt{2}\alpha_1\beta_3 A_a \delta_{-a,b}$
$\alpha_2 A_b$	$\alpha_2\beta_1 A_{-b}$	$b\alpha_2\beta_2 A_a$	$b\alpha_2\beta_3 A_{-a}$
$\alpha_3 A_b$	$\alpha_3\beta_1 A_{-b}$	$a\alpha_3\beta_2 A_a$	$-a\alpha_3\beta_3 A_{-a}$

	$\beta_1 B$	$\beta_2 G_{-b}$	$\beta_3 G_b$
$\alpha_1 B$	$\alpha_1\beta_1 B$	$-b\alpha_1\beta_2 G_{-b}$	$b\alpha_1\beta_3 G_b$
$\alpha_2 A_{-a}$	$\alpha_2\beta_1 A_a$	$(-a)^{(1-b)/2}\alpha_2\beta_2 A_{-a}$	$(-a)^{(1+b)/2}\alpha_2\beta_3 A_{-a}$
$\alpha_3 A_a$	$\alpha_3\beta_1 A_{-a}$	$a^{(1-b)/2}\alpha_3\beta_2 A_a$	$a^{(1+b)/2}\alpha_3\beta_3 A_a$

The structure of the GBV of each fiducial state in the second class is shown in the bottom tabular of Table V, where $G_\pm$ is defined in Eq. (31). Here the sign factors $a, b, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ obey the constraint

$$-ab\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 = 1, \tag{37}$$

and each SIC POVM is specified by three sign functions,

$$h_1 = ab\alpha_1, \quad h_2 = -a\beta_1\beta_2\beta_3, \quad h_3 = b\beta_1. \tag{38}$$

The values of the three sign functions for each group covariant SIC POVM are the same as that in the case of the product basis, see Table IV. In contrast, now fiducial states in the second class rather than in the first class have the same concurrence of $\sqrt{2/5}$, while each fiducial state in the first class may have concurrence of either $\sqrt{(2+2\sqrt{G})/5}$ or $\sqrt{(2-2\sqrt{G})/5}$.

V. SUMMARY

We have studied the structure of HW covariant SIC POVMs in the four-dimensional Hilbert space, in particular, the symmetry transformations within one SIC POVM and among different SIC POVMs. The symmetry group of each HW covariant SIC POVM is shown to be a subgroup of the Clifford group, extending the previous results on prime dimensions [20]. Moreover, we showed that there are 16 additional SIC POVMs by a suitable regrouping of the 256 fiducial states, and demonstrated their equivalence with the original 16 SIC POVMs by establishing an explicit unitary transformation from the original SIC POVMs.

We then revealed the rich structure of these HW covariant SIC POVMs when the four-dimensional Hilbert space is taken as the tensor product of two qubit Hilbert spaces. The introduction of generalized Bloch vectors allowed us to represent the fiducial states and SIC POVMs in a very concise way, and to explore their structure in a systematic manner. In both the product basis and the Bell basis, eight of the 16 SIC POVMs consist of fiducial states with the same concurrence of $\sqrt{2/5}$. They are thus not just symmetric IC POVMs, but supersymmetric IC POVMs.

Acknowledgments

We are grateful for valuable discussions with Markus Grassl. This work is supported by the National Research Foundation and the Ministry of Education, Singapore.

[1] G. Zauner, Ph.D. thesis, University of Vienna, 1999; available online at http://www.mat.univie.ac.at/∼neum/papers/physpapers.html

[2] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, J. Math. Phys. **45**, 2171 (2004).

[3] D. M. Appleby, J. Math. Phys. **46**, 052107 (2005).

[4] A. J. Scott, and M. Grassl, J. Math. Phys. **51**, 042203 (2010).

[5] C. A. Fuchs, arXiv:quant-ph/0205039v1.

[6] A. J. Scott, J. Phys. A: Math. Gen. **39**, 13507 (2006).

[7] P. Delsarte, J. M. Goethals and J. J. Seidel, Philips Res. Rep. **30**, 91 (1975).

[8] M. Grassl, Proceedings of the 2004 ERATO Conference on Quantum Information Science (Tokyo, September, 2004), 60-61; available at arXiv: quant-ph/0406175.

[9] S. G. Hoggar, Geometriae Dedicata **69**, 287 (1998).

[10] M. Grassl, Electronic Notes in Discrete Mathematics **20**, 151 (2005).

[11] M. Grassl, MAGMA 2006 Conference (Technische Universität Berlin, July, 2006); available online at `http://magma.maths.usyd.edu.au/Magma2006/`.

[12] M. Grassl, Seeking SICs: A Workshop on Quantum Frames and Designs (Perimeter Institute, Waterloo, October, 2008); available online at `http://pirsa.org/08100069/`.

[13] M. Grassl, Lect. Notes Comput. Sci. **5393** 89 (2008).

[14] W. K. Wootters, arXiv:quant-ph/0406032 (2004).

[15] D. M. Appleby, H. Dang, and C. Fuchs, arXiv:0707.2071 [quant-ph].

[16] D. M. Appleby, AIP Conf. Proc. **1101** 223 (2009).

[17] P. W. H. Lemmens, J. J. Seidel, J. Algebra **24**, 494 (1973).

[18] D. M. Appleby, S. T. Flammia, and C. A. Fuchs, arXiv:1001.0004 [quant-ph].

[19] D. M. Appleby, arXiv:0909.5233 [quant-ph].

[20] H. Zhu, J. Phys. A: Math. Theor. **43**, 305305 (2010).

[21] H. Kurzweil and B. Stellmacher, *The Theory of Finite Groups, An Introduction* (New York: Springer 2004).

[22] H. Zhu and B.-G. Englert (unpublished).

[23] K. Życzkowski and H. J. Sommers, J. Phys. A: Math. Gen. **34**, 7111 (2001).